Th. Show that the necessary and sufficient for a sequence to be convergent is that it is bounded and has a unique limit point.

Proof: The condition is necessary:

Let the sequence $<x_n>$ is convergent and converges to l.

Then to prove that (i) $<x_n>$ is bounded

(ii) $<x_n>$ has a unique limit point.

As per the theorem that every convergent sequence is bounded

$\Rightarrow \langle x_n \rangle$ is bounded. ---------- (i)

Now let if possible it has two limit points l_1 & l_2 where $l_1 > l_2$.

[Here it should be clear that these l_1 & l_2 are limit points not the limits]

for the sake of convenience let l_1 is the limit also on which $\langle x_n \rangle$ converges.

$\Rightarrow \lim_{n\to\infty} x_n = l_1 \Rightarrow \epsilon > 0$; $\exists\, n_0 \in N$ st.

$$|x_n - l_1| < \epsilon \quad \forall\ n > n_0$$

$\Rightarrow x_n \in (l_1 - \epsilon, l_1 + \epsilon)\ \forall\ n > n_0$. --- (ii)

$\Rightarrow$ In the n.b.d. of l_1, there are infinite members of x_n

$\Rightarrow l_1$ is a limit point of $\langle x_n \rangle$

$\Rightarrow$ from (ii), we have

$$x_n < l_1 + \epsilon \quad \text{for } n > n_0$$

$\Rightarrow x_n < l_1 + \epsilon$ for infinite values of n

$\Rightarrow x_n > l_1 + \epsilon$ for finite values of n

$\Rightarrow x_n < l_1 - \epsilon$ for finite values of n (iii)

Now If $\epsilon = \frac{1}{3}(l_2 - l_1) \Rightarrow l_2 - l_1 = 3\epsilon$

$\Rightarrow l_2 - \epsilon = l_1 + 2\epsilon$

$\Rightarrow l_2 - \epsilon > l_1 + \epsilon$

$\Rightarrow x_n \in (l_2 - \epsilon, l_2 + \epsilon)$ for finite values of n

$\Rightarrow l_2$ is not a limit point of $\langle x_n \rangle$

$\Rightarrow \langle x_n \rangle$ has only one limit point l_1

Condition is sufficient :-

Let $\langle x_n \rangle$ is bounded and it has unique limit point l_1.
We are to prove that $\langle x_n \rangle$ converges to l_1.
Since l_1 is a limit point of $\langle x_n \rangle$
$\Rightarrow$ n.b. of l_1 i.e. $(l_1-\epsilon, l_1+\epsilon)$ contains infinite number of points of $\langle x_n \rangle$
$\Rightarrow$ $x_n \in (l_1-\epsilon, l_2+\epsilon)$ for infinite $n \in N$.
$\Rightarrow$ $x_n \notin (l_1-\epsilon, l_2+\epsilon)$ for finite values of n.
Let n_0 be the largest among these n.
$\Rightarrow$ $x_n \in (l_1-\epsilon, l_2+\epsilon)$ for infinite $n > n_0$
$\Rightarrow$ l_1 is limit of sequence $\langle x_n \rangle$

H.P.

Th. Prove that The necessary and sufficient condition for a monotonically increasing sequence to be convergent is that it is bounded and in that case $\lim_{n\to\infty} x_n = \text{Sup}\{x_n\}$

Proof: The condition is necessary.
Let us take first that $\langle x_n \rangle$ be a monotonic increasing sequence, which convergent and converges to l.
But, we also know, that every convergent

is bounded

$\Rightarrow$ $\langle x_n \rangle$ is bounded. - - - (i)

let $\lim_{n \to \infty} x_n = l$ $\because$ $\langle x_n \rangle$ is convergent.

$\Rightarrow$ for every $\epsilon > 0$, $\exists$ $n_0 \in N$ st. $|x_n - l| < \epsilon$ $\forall n > n_0$

$\Rightarrow$ $l - \epsilon < x_n < l + \epsilon$ $\forall n > n_0$

Since $\langle x_n \rangle$ is monotonically increasing.

$\Rightarrow$ for every $\epsilon > 0$, $l + \epsilon$ is an upper bound of $\langle x_n \rangle$.

$\Rightarrow$ $x_n \geq x_{n_0}$ $\forall n > n_0$

$\Rightarrow$ $x_n < l + \epsilon$ $\forall n > n_0$

Now, $l - \epsilon$ is not an upper bound, we will try to prove that l is also an upper bound of x_n, if so, then it will be the Sup. of $\langle x_n \rangle$. We will prove it through contradiction.

Let us assume that l is not the Sup. of $\langle x_n \rangle$.

$\Rightarrow$ $x_m > l$ for some $m \in N$

$\Rightarrow$ $x_m - l > 0$ for some $m \in N$

let $x_m - l = \delta > 0$

Here $x_m - \delta/2 = l + \delta/2 < x_m$ for some $m \in N$.

if $\epsilon = \delta/2$ (set) $\Rightarrow$ $x_m > l + \epsilon$ for some $m \in N$

$\Rightarrow$ But this is contrary to the statement that $l + \epsilon$ is an upper bound of $\langle x_n \rangle$

$\Rightarrow$ our assumption is wrong

$\Rightarrow$ l is sup of $\langle x_n \rangle$.

$\Rightarrow$ $\sup \langle x_n \rangle = \lim\limits_{n \to \infty} x_n$

H.P.